Nonuniform Flow Through Nozzles

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The jet thrust and the required throat area for a propulsion nozzle may be calculated from the measured stagnation pressure, the stagnation temperature, and the upstream static pressure of the flow entering the nozzle. For flows with nonuniformities in the stagnation quantities and in exit static pressure, integral relations are developed which allow the calculation of increments to thrust and throat area due to the nonuniformities. To illustrate the important nonuniformity interactions, the integral relations are linearized for the special case of uniform γ. As a result, one finds expressions for the correction factors to thrust and area which are the sum of terms that consist of nonuniformity parameters multiplied by sensitivity or influence coefficients. The magnitude of these influence coefficients depends on nozzle parameters and show the relative importance of various kinds of nonuniformities.

I. Introduction

Gas generator methods for the determination of engine thrust depend on the measurement of total pressure and total temperature of the gas flow. This is usually done at a station between the turbomachinery and the exit nozzle where adequate flow quality (nearly rectilinear) is expected at relatively low local Mach number. Typically, the determination of thrust involves averaging so that the one-dimensional equivalent total pressure and temperature can be used to calculate jet exit momentum.

Such averaging methods have varying degrees of applicability and their use results in inaccuracy for thrust and weight flow determination.[1]

In this work, the approach is taken that profiles to total temperature and pressure are measured upstream of a nozzle when the static pressure is known and uniform. With the assumption that the flow distance between the measurement station and the throat or exit are sufficiently short that the profiles are not altered by heat or momentum transfer, each of the stream tubes making up the flow expands isentropically to the exit pressure which itself may be nonuniform.

Values for the average total pressure and average total temperature which result from the insistence on conservation of mass and total enthalpy are of interest, because they can be used in the simple one-dimensional forms of the nozzle flow equations. The most common average used for the determination of thrust is the area-weighted mean because of its simplicity. In the following, the equations for the equivalent (mass, total enthalpy conservation) total pressure and total temperature are derived, highlighting the relationship to the area-weighted quantities. The jet momentum obtained at the nozzle exit will be calculated for the nonuniform flow. Since total mass flow and total enthalpy flux involve both total temperature and pressure, the averaging is coupled and not generally reducible to simple averages of pressure and temperature obtained separately. In order to illustrate this coupling, the equations are specialized to a slightly nonuniform flow, permitting linearization of the equations.

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II. Analysis

The analysis of Bernstein, Heiser, and Hevenor[2] is extended to handle "smoothly" varying profiles of total pressure and temperature. We consider the flow to consist of a large number of noninteracting streamtubes for each of which the total pressure, total temperature, and the flow area have been measured. The profiles could presumably arise from boundary layers, wakes, nonuniform work input by rotors, nonuniform efficiency, heat losses, probes, etc.

In the first part of the development, gross flow curvature effects will be neglected so that uniform static pressure at any axial position may be assumed. This assumption will not be made in Sec. IV. In Fig. 1 the flow geometry is shown. At station 1 the total pressure and temperature measurements are made so that $p_t(A_1)$ and $T_t(A_1)$ are known in addition to p_1. The continuity equation for any one streamtube is

$$
\begin{aligned}
\mathrm{d}\dot{m} &= \frac{p_1 \mathrm{d}A_1}{\sqrt{[(\gamma-1)/2\gamma]RT_t(A_1)}}\left(\frac{p_t(A_1)}{p_1}\right)^{(\gamma-1)/\gamma} \\
&\quad\times\left[1-\left(\frac{p_1}{p_t(A_1)}\right)^{(\gamma-1)/\gamma}\right]^{1/2} \\
&= \frac{p_2 \mathrm{d}A_2}{\sqrt{[(\gamma-1)/2\gamma]RT_t(A_1)}}\left(\frac{p_t(A_1)}{p_2}\right)^{(\gamma-1)/\gamma} \\
&\quad\times\left[1-\left(\frac{p_2}{p_t(A_1)}\right)^{(\gamma-1)/\gamma}\right]^{1/2}
\end{aligned}
\tag{1}
$$

where the subscript 2 applies to any point in the nozzle. The latter portion of this equation gives the streamtube area ratio in terms of the pressure.

Choking Condition

The choking condition for nonuniform flow may be stated[2] as

$$\int_0^{A^*}\frac{\mathrm{d}A^*}{\gamma}\left(\frac{1}{M^{*2}}-1\right)=0 \tag{2}$$

The Mach number in the throat, M^*, is given by

$$
\begin{aligned}
M^{*2} &= \frac{2}{\gamma-1}\left[\left(\frac{p_t(A_1)}{p^*}\right)^{(\gamma-1)/\gamma}-1\right] \\
&= \frac{2}{\gamma-1}\left[\left(\frac{p_t(A_1)}{p_1}\right)^{(\gamma-1)/\gamma}\left(\frac{p_1}{p^*}\right)^{(\gamma-1)/\gamma}-1\right]
\end{aligned}
\tag{3a}
$$

Since the total pressure and static pressure often appear in a ratio to the $(\gamma-1)/\gamma$ power, it is convenient to define a new measure of the total pressure distribution

$$z_i = (p_t(A_1)/p_i)^{(\gamma-1)/\gamma} \qquad (3b)$$

where i refers to any station in the nozzle. Futhermore, it is convenient to normalize the transverse (if two-dimensional) or radial area coordinate A_1 where the measurement $p_t(A_1)$ is made by the total flow area at that station $A_{1,\,\mathrm{total}}$. Thus, let $x_1 = A_1/A_{1,\,\mathrm{total}}$, and Eq. (3a) becomes

$$M^{*2} = \frac{2}{\gamma-1}[\alpha^* z_1(x_1) - 1] \qquad \alpha^* \equiv \left(\frac{p_1}{p^*}\right)^{(\gamma-1)/\gamma} \qquad (3c)$$

In the following, a nonuniformity in γ will not be considered because of the algebraic complexity and because physical situations where p_t and T_t are nonuniform often involve the same gas. In cases where gases with differing γ's are expanded, the multiple stream analysis of Ref. 2 is appropriate. The continuity equation, Eq. (1), thus becomes

$$\frac{d\dot m}{dA_1} = \frac{p_1}{\sqrt{[(\gamma-1)/2\gamma]RT_t(A_1)}}\sqrt{z_1^2 - z_1}$$

$$= \frac{p_2}{\sqrt{[(\gamma-1)/2\gamma]RT_t(A_1)}}\sqrt{(\alpha_2 z_1)^2 - \alpha_2 z_1}\,\frac{dA_2}{dA_1} \qquad (4)$$

With station 2 specialized to the throat, the latter portion of this equation gives dA^* in terms of dA_1 so that the choking criterion Eq. (2), may be written as

$$\int_0^1 \frac{(z_1-1)^{1/2}}{(\alpha^* z_1 - 1)^{3/2}}\,dx_1 = \frac{2}{\gamma-1}\int_0^1 \frac{(z_1-1)^{1/2}}{(\alpha^* z_1 - 1)^{1/2}}\,dx_1 \qquad (5a)$$

This equation gives p^*. If the measurement station is at the throat $p^* = p_1$ (or $\alpha^* = 1$) and Eq. (5a) becomes

$$\int_0^1 \frac{1}{z_1 - 1}\,dx_1 = \frac{2}{\gamma - 1} \qquad (5b)$$

If the flow is not choked, the use of the choking condition is not appropriate and Eq. (4) merely gives the flow area A_2 required for a given p_2 and the nonuniformity in p_t that is,

$$\int dA_2 = \alpha_2^{(\gamma+1)/2(\gamma-1)}\int \frac{\sqrt{z_1 - 1}}{\sqrt{\alpha_2 z_1 - 1}}\,dA_1$$

In the following, the assumption of choked flow will be made. Accurate thrust determination is primarily of interest in those cases, although extension of this work is straightforward.

Equivalent Average Total Pressure and Temperature

The customary equation for mass flow through a choked nozzle, stated in terms of total temperature and pressure, viz.,

$$\int_0^{A_1}\frac{d\dot m}{dA_1}\,dA_1 = \dot m = \frac{p_{te}}{\sqrt{RT_{te}}}\sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)} A^* \qquad (6)$$

and the definition of mass-averaged total enthalpy

$$C_p T_{te} = \frac{1}{\dot m}\int_0^{A_1} C_p T_t(A_1)\frac{d\dot m}{dA_1}\,dA_1 \qquad (7)$$

may be used to define equivalent average total pressure and average total temperature of the flow through the nozzle. The term $d\dot m/dA_1$ in Eqs. (6) and (7) is obtained from Eq. (4).

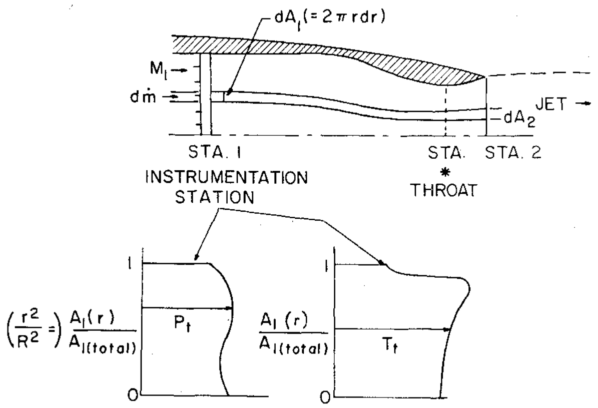

Fig. 1 Schematic of the nozzle flow showing nonuniformity profiles in total pressure and total temperature.

With the definition

$$\tau_1 = RT_{t_1}(x_1)/RT_{t\mathrm{ref}} \qquad \tau_e = RT_{te}/RT_{t\mathrm{ref}} \qquad (8)$$

Here $(RT_t)_{\mathrm{ref}}$ may be any useful value of that quantity, such as the value at $x_1 = 0$.

Eq. (7), for the unknown T_{te} becomes

$$\tau_e = \frac{\int_0^1 \sqrt{\tau_1}\,(\sqrt{z_1(z_1-1)})\,dx_1}{\int_0^1 (1/\sqrt{\tau_1})\sqrt{z_1(z_1-1)}\,dx_1} \qquad (9)$$

To obtain the value of the average stagnation pressure p_{te}, Eq. (6) is used

$$\frac{p_{te}}{p_1} = \left(\frac{2}{\gamma-1}\right)^{1/2}\left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/2(\gamma-1)} \times \int_0^1 \sqrt{\frac{\tau_e}{\tau_1}}\sqrt{z_1(z_1-1)}\,dx_1\frac{A_1}{A^*} \qquad (10)$$

The ratio A_1/A^* is obtained from Eq. (4):

$$\frac{A^*}{A_1} = (\alpha^*)^{\gamma/(\gamma-1)}\int_0^1 \frac{z_1\sqrt{1-1/z_1}}{\alpha^* z_1\sqrt{1-1/\alpha^* z_1}}\,dx_1 \qquad (11)$$

Thus, Eq. (10) becomes

$$\frac{p_{te}}{p_1} = \left(\frac{2}{\gamma-1}\right)^{1/2}\left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/2(\gamma-1)}(\alpha^*)^{-\gamma/\gamma-1} \times \frac{\int_0^1\sqrt{\frac{\tau_e}{\tau_1}}\sqrt{z_1(z_1-1)}\,dx_1}{\int_0^1 \frac{\sqrt{z_1(z_1-1)}}{\sqrt{\alpha^* z_1(\alpha^* z_1 - 1)}}\,dx_1} \qquad (12)$$

The equivalent total pressure p_{te} may also be written in terms of the area-weighted average total pressure $\bar p_t$, which is

$$\frac{\bar p_t}{p_1} = \int_0^1 z_1^{\gamma/(\gamma-1)}\,dx_1 \qquad (13)$$

Thus, Eq. (12) becomes

$$\frac{p_{te}}{\bar p_t} = \frac{\sqrt{\frac{2}{\gamma-1}}\left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/2(\gamma-1)}}{\int_0^1 (\alpha^* z_1)^{\gamma/(\gamma-1)}\,dx_1}\;\frac{\int_0^1\sqrt{\frac{\tau_e}{\tau_1}}\sqrt{z_1(z_1-1)}\,dx_1}{\int_0^1\frac{\sqrt{z_1(z_1-1)}}{\sqrt{\alpha^* z_1(\alpha^* z_1-1)}}\,dx_1} \qquad (14)$$

which, in the limit of z_1 independent of x_1, reduces to the trivial result $p_{te} = \bar{p}_t$ since $\alpha^* z_1 = (\gamma+1)/2$ in that case.

The momentum carried by each stream tube upon expansion to pressure p_2 is

$$dJ = (\rho u^2)_2 dA_2 = \gamma p_2 M_2^2 \frac{dA_2}{dA_1} dA_1 \quad (15)$$

Following the method used to arrive at Eq. (3), the flow Mach number at station 2 is given by

$$M_2^2 = \frac{2}{\gamma-1}[\alpha_2 z_1 - 1] \qquad \alpha_2 \equiv (p_1/p_2)^{(\gamma-1)/\gamma} \quad (16)$$

Thus, the total integrated momentum flux is

$$J = \gamma p_2 A_1 \frac{2}{\gamma-1}\int_0^1 (\alpha_2 z_1 - 1)\left(\frac{p_1}{p_2}\right)^{1/\gamma} \frac{(1-1/z_1)^{1/2}}{(1-1/\alpha_2 z_1)^{1/2}} dx_1$$

$$= \frac{2\gamma}{\gamma-1} p_1 A_{1\,\mathrm{tot}} \alpha_2^{-1/2} \int_0^1 \sqrt{(\alpha_2 z_1 - 1)(z_1 - 1)}\, dx_1 \quad (17)$$

which is valid for choked and unchoked flow.

Limit of Uniform Flow

In the limit of uniform flow, i.e., z_1 independent of x_1, the integrals become trivial. The choking condition becomes

$$p_t/p^* = (\alpha^* z_1)^{\gamma/(\gamma-1)} = [(\gamma+1)/2]^{\gamma/(\gamma-1)} \quad (18)$$

The momentum at the nozzle exit is

$$\frac{J}{p_t A^*} = \gamma\left(\frac{2}{\gamma-1}\right)^{1/2}\left(\frac{\gamma+1}{2}\right)^{-(\gamma+1)/2(\gamma-1)}\left(1-\left(\frac{p_2}{p_t}\right)^{(\gamma-1)/\gamma}\right)^{1/2} \quad (19)$$

III. Almost Uniform Flows

The preceding expressions (Eqs. 9, 11, 14, and 17) give the relationship between the equivalent averages and those obtained by the flow area weighing method for specific (measured) variations in total pressure and temperature. To see the importance of the various kinds of nonuniformities and their interactions, it is useful to develop algebraic relations for the correction factors between equivalent and area-weighted averages in terms of the nonuniformity profile shapes. Thus, for almost uniform flows, it may be appropriate to let total pressure, for example, be given by

$$p_t(x_1)/p_1 = a^{\gamma/(\gamma-1)}(1+f(x_1)) \quad (20)$$

where the area variation of p_t is lumped into $f(x_1)$. If the integral

$$\int_0^1 f(x_1)\, dx_1 = 0$$

then the constant $a^{\gamma/(\gamma-1)}$ may be interpreted as the ratio of area-weighted average pressure, $\bar{p}_t$ to p_1. The exponent is written in this form for algebraic convenience. Thus,

$$\bar{p}_t/p_1 = a^{\gamma/(\gamma-1)} \quad (21)$$

Similarly, the temperature variation may be written

$$T_t(x_1)/T_{t\mathrm{ref}} = 1 + g(x_1) \quad (22)$$

If the pressure and temperature variations are small, i.e., $f,\ g \ll 1$, Eqs. (20) and (22) may be used to obtain z_1 and τ_1 required in the integral Eqs. (5, 9, 11, 14, and 17). Carrying all terms up to the second order, the indicated integrations will involve $\int_0^1 f\, dx$ and $\int_0^1 g\, dx$ which are zero by definition of the area-weighted mean and integrals of the form

$$I_{ff} = \int_0^1 f^2 dx_1 \quad I_{gg} = \int_0^1 g^2 dx_1 \quad I_{fg} = \int_0^1 fg\, dx_1$$

The quantities resulting from the integration will be of the form, for example,

$$P_{te}/P_t = 1 + b_1 I_{ff} + b_2 I_{fg} + b_3 I_{gg}$$

The magnitudes of the coefficients of these integrals determine the relative importance of one kind of uniformity vis-a-vis another. Higher order integrals are neglected because f,g are assumed sufficiently small.

With Eq. (20), z_1 and $z_1 - 1$ become

$$z_1 = a(1 + \Gamma_1 f + \Gamma_2 f^2) \quad (23a)$$

$$z_1 - 1 = (a-1)\left(1 + \Gamma_1 \frac{a}{a-1} f + \Gamma_2 \frac{a}{a-1} f^2\right) \quad (23b)$$

where $\Gamma_1 = (\gamma-1)/\gamma$ and $\Gamma_2 = -(\gamma-1)/2\gamma^2$. In order for the expansion to be valid we must have, from Eq. (23)

$$\left|f\frac{\Gamma_2}{\Gamma_1}\right| = \frac{1}{2\gamma} f \ll 1$$

and

$$\Gamma_1 \frac{a}{a-1} f \ll 1$$

Since $a = (\bar{p}_t/p_1)^{(\gamma-1)/\gamma} = 1 + \bar{M}_1^2(\gamma-1)/2$, this condition becomes

$$\frac{\gamma-1}{\gamma}\left[1 + \frac{2}{\gamma-1}\frac{1}{\bar{M}_1^2}\right] f \ll 1$$

This implies that the extent of the nonuniformity f for which this analysis is valid increases with increasing average duct Mach number $\bar{M}_1$.

The choking criterion involves only pressures, hence only terms involving f. The terms required in Eq. (5) are, in part,

$$(z_1 - 1)^{1/2} = (a-1)^{1/2}\left[1 + \frac{\Gamma_1}{2}\frac{a}{a-1} f + \left(\frac{\gamma-1}{2}\frac{a}{a-1} + 1\right)\right.$$

$$\left.\times \frac{\Gamma_2}{2}\frac{a}{a-1} f^2\right] \quad (24a)$$

and

$$(\alpha^* z_1 - 1)^{-1/2} = (\alpha^* a - 1)^{-1/2}\left[1 - \frac{\Gamma_2}{2}\frac{\alpha^* a}{\alpha^* a - 1} f\right.$$

$$\left. - \left(2\Gamma_2 - \frac{3}{2}\frac{\alpha^* a}{\alpha^* a - 1}\Gamma_1^2\right)\frac{1}{4}\frac{\alpha^* a}{\alpha^* a - 1} f^2\right] \quad (24b)$$

Substituting Eqs. (24) into the choking criterion, Eq. (5a) yields an expression for $\alpha^* a$

$$\alpha^* a = \left(\frac{p_t}{p^*}\right)^{(\gamma-1)/\gamma} = \frac{\gamma+1}{2}\left\{1 + \frac{\gamma-1}{\gamma^2}\left[\left(\gamma + \frac{3}{2}\right)\right.\right.$$

$$\left.\left. - \left(\frac{1}{\bar{M}_1^2} - 1\right)\right] I_{ff}\right\}$$

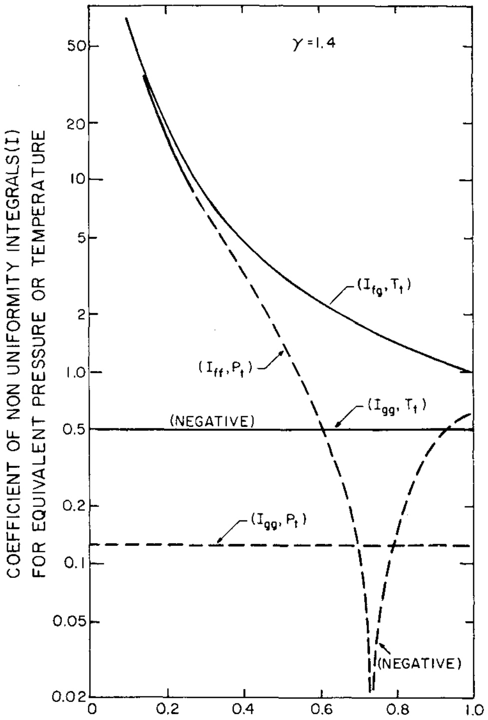

Fig. 2 Variations of the coefficients of the nonuniformity integrals for total pressure (labeled p_t, ———) and for total temperature (labeled T_t,—).

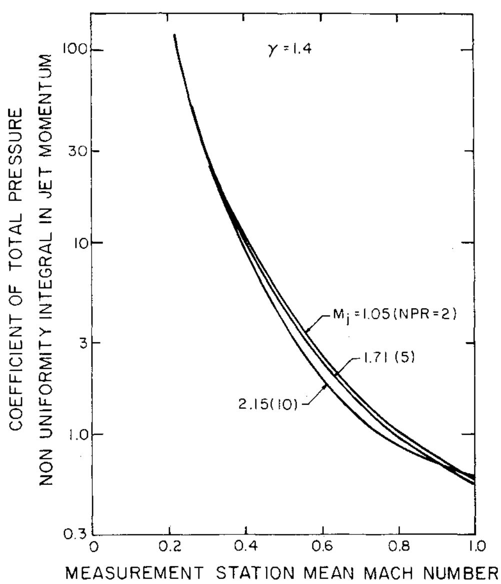

Fig. 3 Variation of the coefficients of the nonuniformity integral I_{ff} for jet momentum with M_l and M_j [see Eq. (29)].

or

$$\frac{p^*}{p^*_{\text{ideal}}} = 1 - \frac{1}{\gamma}\left[\left(\gamma + \frac{3}{2}\right) - \left(\frac{1}{\bar{M}_l^2} - 1\right)\right] I_{ff} \tag{25}$$

Note that the coefficient of $I_{ff} = 0$ when $\bar{M}_l = 0.506$, $\gamma = 1.4$.

The average temperature is obtained by substituting Eqs. (20) and (22) into Eq. (9):

$$\tau_e = \frac{T_{te}}{\bar{T}_t} = 1 + \left[1 + \frac{1}{\gamma}\left(\frac{1}{\bar{M}_l^2} - 1\right)\right] I_{fg} - \frac{1}{2} I_{gg} \tag{26}$$

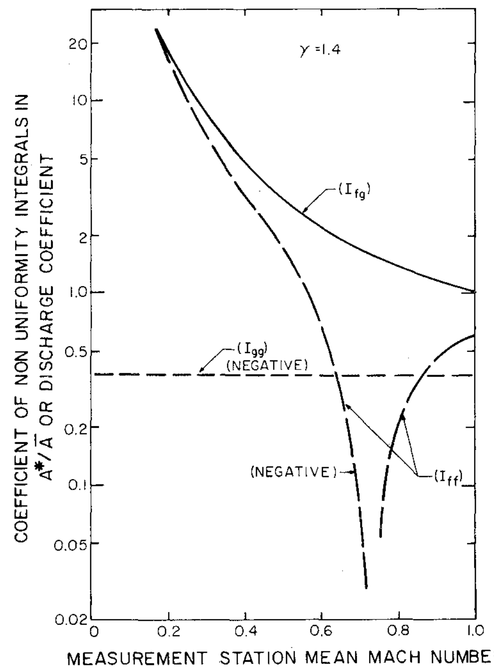

Fig. 4 Variation of the coefficients of the nonuniformity integrals for discharge coefficient with M_l [see Eq. (31)].

At values of $\bar{M}_l$ at about 0.5, which is typical of the upstream Mach number of nozzle flows, the magnitude of the I_{fg} coefficient is about 3, indicating that the coupling of total pressure and total temperature nonuniformities plays a greater role in the approximation of T_{te} than the temperature nonuniformity alone. The total pressure nonuniformity plays no role by itself, as noted from the absence of I_{ff} in this expression.

Similarly,

$$\frac{p_{te}}{\bar{p}_t} = 1 + \frac{1}{\gamma}\left[\left(\frac{1}{\bar{M}_l^2} - 1\right) - \frac{\gamma + 1}{2\gamma}\right] I_{ff} + \frac{1}{8} I_{gg} \tag{27}$$

For $\bar{M}_l \sim 0.5$, the coefficient of $I_{ff} \sim 1.5$, which emphasizes that the role played by the temperature nonuniformity is minor in determining p_{te}. Even in the absence of a temperature nonuniformity, we note that the error made in approximating p_{te} by $\bar{p}_t$ increases with decreasing flow Mach number at the measurement station ($\bar{M}_l$). Note the absence of any I_{fg} term which reflects coupling of the nonuniformity profiles. The coefficients multiplying the various nonuniformity integrals in Eqs. (26) and (27) are plotted in Fig. 2 as function of upstream Mach number $\bar{M}_l$.

The momentum of the jet may be obtained from the general Eq. (17), assuming the static pressure at the exit to be uniform. This limits the analysis to nozzles with "small" convergence or divergence angles. This limitation will be discussed at the conclusion of this section.

Expansion of the flow involves the nozzle pressure ratio $p_t/p_2 = [(\alpha_2 z_l)^{\gamma/(\gamma-1)}]$. The mean total pressure may be used to define a mean jet Mach number as

$$\bar{p}_t/p_2 = (\alpha_2 a)^{\gamma/(\gamma-1)} = (1 + (\gamma-1)M_j^2/2)^{\gamma/(\gamma-1)} \tag{28}$$

Thus, substituion of Eqs. (16) and (18) into Eq. (14) and performing the integrations yields

$$\frac{J}{J_{\text{ideal}}} = 1 - \frac{1}{2\gamma^2}\left[(\gamma - 1) + \left(\frac{1}{M_l^2} + \frac{1}{M_j^2}\right) + \left(\frac{1}{M_l^2} - \frac{1}{M_j^2}\right)^2\right] I_{ff} \tag{29}$$

Note the absence of terms involving the temperature nonuniformity. Figure 3 shows the importance of the pressure nonuniformity with nozzle pressure ratio and measurement station flow Mach number. For a small value of $\bar{M}_1$, the nonuniformity is very important.

The mass flow rate, which is known in terms of p_{te} and T_{te}, can also be expressed in terms of $\bar{p}_t$ and $\bar{T}_t$. Equating these mass flows gives the ratio of the true sonic area required, A^*, in terms of the area that would have been calculated if area-weighted means were used, $\bar{A}^*$.

$$\frac{p_{te}}{\sqrt{RT_{te}}} A^* = \frac{\bar{p}_t}{\sqrt{R\bar{T}_t}} \bar{A}^* \tag{30}$$

Invoking the expressions for $p_{te}/\bar{p}_t$ and $T_{te}/\bar{T}_t$ gives

$$\frac{A^*}{\bar{A}^*} = 1 + \frac{1}{4}\left[1 + \frac{1}{\gamma}\left(\frac{1}{\bar{M}_1^2} - 1\right)\right] I_{fg} - \frac{3}{8} I_{gg} - \frac{1}{\gamma}\left[\left(\frac{1}{\bar{M}_1^2} - 1\right) - \frac{\gamma+1}{2\gamma}\right] I_{ff} \tag{31}$$

Figure 4 shows the coefficients of the nonuniformity integrals. This ratio may be viewed as a nozzle discharge coefficient associated with the nonuniformities. Since the nozzle may be properly sized by the preceding procedure using the measurements taken at station 1, it is implicit that the mass flow is known and thus the ratio of momenta, as in Eq. (29) may be interpreted as a thrust coefficient associated with the nonuniformities.

IV. Nonuniform Static Pressure at the Nozzle Exit

Many types of nozzle configurations may be cited in which internal and/or external flows are such that the static pressure distribution is nonuniform in the plane of the measured nozzle areas. Typically, for convergent nozzles the pressures of streamtubes further toward the flow centerline are higher due to the flow curvature as shown in Fig. 5. Means are available[3,4] where the static pressure profile at the nozzle exit may be determined for a uniform flow. Taking the approach that nonuniformity effects do not influence the static pressure distribution to first order, the impact of the exit static pressure profile may be calculated by generalizing the momentum equation to include a variation in $p_2(x)$.

It is thus assumed that

$$p_2 = p_{2_{\text{ref}}}[1 + h(x_2)] \tag{32}$$

where $p_{2_{\text{ref}}}$ is a reference exit static pressure. The momentum flux is obtained from Eq. (17), modified to retain the pressure p_2 in the integral. Thus,

$$J = \frac{2\gamma}{\gamma-1} p_1 A_{1,\text{total}} \int_0^1 \frac{1}{\sqrt{\alpha_2}} \sqrt{(\alpha_2 z_1 - 1)(z_1 - 1)}\, dx_1 \tag{33}$$

where

$$\alpha_2 = (p_1/p_2)^{(\gamma-1)/\gamma}$$

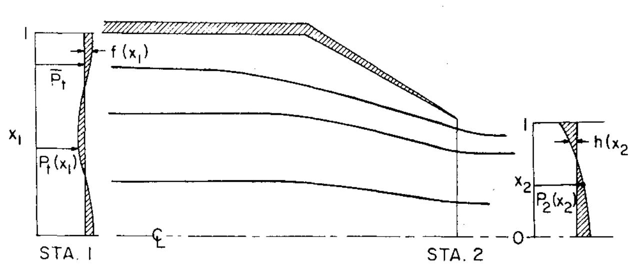

Fig. 5 Schematic of nozzle showing the nonuniform exit static pressure distribution $h(x_2)$.

and is assumed known as a function of x_2. The quantity z_1 reflects only the varying total pressure at x_1. Equation (33) can therefore be integrated when the functional dependence of x_2 on x_1 is known.

The continuity equation (4) relates the differential streamtube sizes at nozzle stations 1 and 2:

$$p_1 A_{1,\text{total}} dx_1 = p_2(x_2) A_{2,\text{total}} dx_2 \sqrt{\frac{\alpha_2(x_2)[\alpha_2(x_2) z_1(x_1) - 1]}{z_1(x_1) - 1}} \tag{34}$$

or

$$\frac{A_{1,\text{total}}}{A_{2,\text{total}}} dx_1 = dx_2 \alpha_2^{-(\gamma/(\gamma-1) - 1/2)} \sqrt{\frac{\alpha_2 z_1 - 1}{z_1 - 1}}$$

with $\bar{\alpha}_2 = (p_1/p_{2\text{ref}})^{(\gamma-1)/\gamma}$

$$\alpha_2 = \bar{\alpha}_2[1 + h(x_2)]^{-(\gamma-1)/\gamma} \quad z_1 = a(1 + \Gamma_1 f + \Gamma_2 f^2) \tag{35}$$

substituted into Eq. (34), retaining first-order terms in f,h, and integrating from zero to x_1, x_2, respectively, yields

$$\frac{x_1}{x_2} = 1 + \frac{1}{\gamma}\left[\left(1 - \frac{1}{\bar{M}_1^2}\right) I_h - \left(\frac{1}{\bar{M}_1^2} - \frac{1}{\bar{M}_j^2}\right) I_f\right] \tag{36}$$

The integrals I_f and I_h are indefinite and defined as

$$I_f = \frac{1}{x_1} \int_0^{x_1} \xi f(\xi)\, d\xi$$

$$I_h = \frac{1}{x_2} \int_0^{x_1} \xi h(\xi)\, d\xi \tag{37}$$

With I_f, I_h very small, i.e., nearly uniform p_t and p_2, the equation gives a unit-stretching function. Note, both nonuniformities in p_t and p_2 introduce nonlinearity. Thus one might expect $h(x_2)$ to be transformed to $h(x_1)$, as shown in Fig. 6. If the measurement station were at the exit of a choked convergent nozzle, $\bar{\alpha}_2 = 1$ and $\bar{M}_1 = 1$, then the coefficients of the terms I_f, I_h vanish and the trivial result of unit stretching is obtained.

One may, therefore, take the static pressure nonuniformity as a known function of x_1 and Eq. (33) may thus be integrated to obtain the exit momentum. One obtains, after integrating over x_1

$$J = \frac{2\gamma}{\gamma-1} p_1 A_{1,\text{total}} \bar{\alpha}_2^{-1/2} \sqrt{(\bar{\alpha}_2 a - 1)(a-1)} \times \{1 + C_{ff} I_{ff} + C_{fh} I_{fh} + C_{hh} I_{hh}\} \tag{38}$$

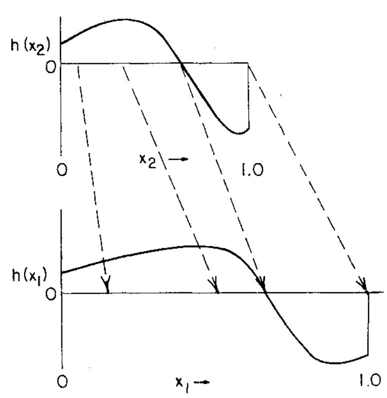

Fig. 6 Stretching of the exit static pressure profile. Schematic representation of Eq. (36).

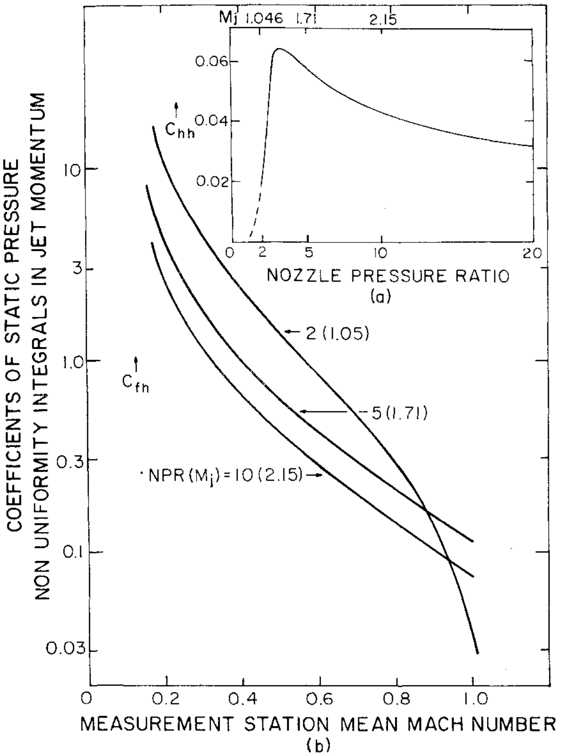

Fig. 7 Coefficients of nonuniformity integrals for jet momentum with nonuniform exit static pressure [see Eq. (38)].

In obtaining this expression in this form, the integral of h was taken as zero. This fixes $p_{2_{ref}}$ as being the area-weighted mean value. The integrals I_{fh} and I_{hh} are defined as

$$I_{fh} = \int_0^1 fh \, dx_l \qquad I_{hh} = \int_0^1 h^2 \, dx_l \tag{39}$$

The C_{ij} coefficients are given by

$$C_{fh} = -\frac{1}{\gamma^2 \bar{M}_j^2}\left\{\frac{1}{\bar{M}_l^2} - \frac{1}{\bar{M}_j^2}\right\}$$

$$C_{hh} = \frac{1}{2\gamma^2}\frac{1}{\bar{M}_j^2}\left\{1 - \frac{1}{\bar{M}_j^2}\right\}$$

$$C_{ff} = \frac{1}{2\gamma^2}\left[(\gamma - 1) + \left(\frac{1}{M_l^2} + \frac{1}{M_j^2}\right) + \left(\frac{1}{M_l^2} - \frac{1}{M_j^2}\right)^2\right] \tag{40}$$

C_{ff} is identical to that obtained in Eq. (29) and is plotted in Fig. 3.

Figure 7a shows the variation of C_{hh} with nozzle pressure ratio which is related to M_j by Eq. (28). Note that the impact of the static pressure nonuniformity is relatively small, peaking with an influence coefficient of 0.06 in the range where turbojets and low bypass fans are operated at cruise. Figure 7b shows C_{fh} as a function of M_l. The magnitude of C_{fh}, if relatively large at low M_l as are many of the coefficients described earlier, increasing for low jet Mach number or nozzle pressure ratio.

V. Concluding Remarks

In this paper, the important nonuniformities affecting mass average total pressure, total temperature, and jet exit momentum have been identified. In the range of $M_l \simeq 0.5$, commonly the value upstream of nozzles, these are: 1) for total temperature—the coupled nonuniformities in pressure and temperature, I_{fg}; 2) for total pressure—the nonuniformity in pressure only, I_{ff}; 3) for jet momentum—the nonuniformity in pressure only, I_{ff}. The discharge coefficient's dependence on nonuniformities is largely due to pressure I_{ff} and the combination of pressure and temperature I_{fg}.

Nozzle gross thrust is calculated from the sum of the momentum term and $(p_{exit} - p_{ambient})A_{exit}$. In this paper, only the jet momentum term is discussed. It should be noted that the exit pressure should be the area-weighted average value with I_{hh} making no contribution to this thrust component.

The effect of nonuniformity in exit static pressure plays a moderately important role on the momentum component of thrust only through coupling with the total pressure distribution. The exit static pressure nonuniformity is, by itself, not important ($C_{hh} \ll 1$) in affecting the momentum term of gross thrust.

This analysis clearly underestimates the magnitude of losses due to nonuniformities because the flow is assumed isentropic. Adjacent streamlines do interact in real flows and, as a result of the entropy generation, additional total pressure and losses will be encountered.

The equations for equivalent total pressure and temperature and for jet momentum can, nevertheless, be easily obtained from the integrations indicated in the text to arrive at first-order corrections which should be applied to the thrust calculation procedure using the gas generator method.

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